EXTENSION SETS

BY
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1. **Historical introduction.** In a symposium lecture in 1932 R. L. Wilder [10](1), in discussing the structure of continua, indicated the possibility of extending G. T. Whyburn's cyclic element theory by making use of combinatorial methods. The notion of an "nth order cyclic element" was developed in a paper published by Whyburn [8] some two years later. In this paper Wilder's suggestion will be modified in replacing "combinatorial methods" by "analytic methods." Considerable dependence will be placed on the ideas and techniques formulated by C. Borsuk, S. Eilenberg and W. Hurewicz. It is perhaps worthy of remark that Wilder does not refer to the third "branch" of topology, analytic topology, in this lecture though it played a prominent part in another of his lectures given about six years later [11].

It will be recalled that an A-set is a retract of its containing space, so that, guided by Borsuk's result characterizing absolute retracts as those spaces which have the extension property, we are led to the notion of an extension set which plays the part of an A-set. However an extension set is a relative and not an absolute invariant because, by necessity, it must be intimately related to its containing space.

Section two will be devoted to results of a preliminary nature, mainly existence theorems. In the next section we consider sets that are "trivial" in dimension n; more exactly—sets which are hereditary relative to the property of admitting no essential transformations into the n-sphere. For separable spaces some of the results here are immediate consequences of well known theorems of dimension theory. The notion of an extension set is introduced in the fourth section. These sets have many of the properties possessed by A-sets in a locally connected continuum. Thus the intersection of any family of extension sets is again an extension set. Also our "cyclic elements" are extension sets. With the addition of a separation axiom it is shown that an extension set in one dimension is also an extension set of any higher dimension. The notion of a general "endelement" is introduced and also shown to be an extension set. The study of these sets is continued in section five where other results are formulated. Scattered through the paper are theorems concerning "continua of order n" and "endelements of order n."

While no direct use will be made of their results it is necessary to call attention to the work of several mathematicians who have studied and gen-

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(1) Numbers in brackets refer to the Bibliography at the end of the paper.

eralized the cyclic element theory: W. L. Ayres, V. W. Adkisson, G. E. Albert, D. W. Hall, F. B. Jones, J. L. Kelley, C. Kuratowski, Saunders MacLane, R. L. Moore, T. Radó, P. V. Reichelderfer, Hassler Whitney and J. W. T. Youngs. References to their papers will be found in [7] and [9].

References should also be made to papers by J. Rozanska [6] and C. Borsuk [12] containing concepts and results which are related to ideas and theorems formulated here.

I am glad to acknowledge that my interest in non-separable spaces and in homotopy theory springs from many discussions with S. Lefschetz.

2. Mathematical introduction. While many of our results are valid in a more general situation an optimum degree of generality will consistently be attained if we assume that *H is a compact Hausdorff space*. Here *compact* replaces the more unwieldy "bicompact." In the further interests of verbal simplicity we use *subspace* for "closed subset of H."

The space H is, of course, normal. We introduce the additional separation axiom as follows: A topological space F is said to be of *type* V if for any pair of closed subsets R_1 and R_2 there exists a decomposition of F into closed sets F_1 and F_2 containing R_1 and R_2 such that $F_1 \cdot F_2 \cdot (R_1 + R_2) = R_1 \cdot R_2$. We are unable to give a reference to the occurrence of this condition as an axiom though it is a well known property of metric spaces. In a later paper it will be necessary to consider spaces in which each subset has the above property. When H is assumed to be of type V attention will be called to this postulate.

We denote by S a compact nondegenerate Hausdorff space which has the *neighborhood extension property*: Any mapping of a closed set A of a normal space into S can be extended to a mapping of a neighborhood U of A into S. When the spaces involved are separable metric then the neighborhood extension property is equivalent to the property of being an absolute neighborhood retract as Borsuk [4, p. 60] has shown.

The words "mapping" and "transformation" are used in the sense of "continuous correspondence" and "into" and "onto" have their usual meaning. Two transformations f, g into S are said to be homotopic (in symbols $f \sim g$) if there is a mapping h defined for all points x under consideration and all $t \in (01)$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. The values of h are assumed to lie always in S. No confusion is to be feared concerning the symbol $\sim$ since we do not use homology in this paper. In most of our results S will be taken to be an n-sphere S_n. We need the following result of Borsuk as extended by Dowker [3, p. 86]:

(2.1) *If f maps H into S and g maps the subspace X into S and $f \sim g$ on X then g has an extension $\bar{g}: H \rightarrow S$ such that $f \sim \bar{g}$ on H.*

A mapping f of a topological space F into S is said to be *inessential* if it is homotopic to a mapping g such that $g(F)$ is a proper subset of S. Also f is null-homotopic (in symbols, $f \sim 0$) if f is homotopic to a transformation g

for which $g(F)$ is a point of S. It is obvious that an inessential mapping into an n-sphere is null-homotopic.

(2.2) *If f and g are mappings of H into S and $f\sim g$ on the subspace X then $f\sim g$ on some open set containing X.*

Proof. Let h be a transformation of $X\times(01)$ into S such that $h(x, 0)=f(x)$ and $h(x, 1)=g(x)$. We may extend the definition of h by putting $h=f$ on $X\times(0)$ and $h=g$ on $X\times(1)$. Then $K=H\times(0)+X\times(01)+H\times(1)$ is a closed set in the cartesian product of H and (01). In virtue of the fact that S has the neighborhood extension property, we may suppose that h has been extended to a mapping of a neighborhood U of K. In view of a lemma of Dowker's [3, p. 86] there is an open set V containing X and for which we have the inclusion $V\times(01)\subset U$. Then clearly f and g are homotopic on the set V.

(2.3) *If f is a transformation of a subspace Y into S and $f\sim 0$ on the subspace $X\subset Y$ then $f\sim 0$ on an open subset of Y containing X.*

This follows from (2.2) and is a generalization of a result due to S. Eilenberg [9, p. 221].

The term *ordered*, as applied to a family of subspaces, will mean that, of any pair, one is a subset of the other.

A property P admissible for subspaces will be termed *inductive* if, when each of an ordered family of subspaces has property P, then their intersection also has property P. Many writers have formulated results closely related to the following which we call simply the "Brouwer induction theorem." References will be found in [7, p. 488].

(2.4) *If P is an inductive property and the subspace X has P then there exists a subspace contained in X and minimal relative to having property P.*

A mapping f of the subspace X into S is said to be *irreducibly essential* on X provided that f is essential on X but inessential on any subspace properly contained in X.

(2.5) *An essential transformation of H into S is irreducibly essential on some subspace.*

Proof. If we refer to (2.4) it is enough to show that, if P is the property of being a subspace on which the mapping $f:H\rightarrow S$ is essential, then P is an inductive property. To this end let $[X]$ be an ordered family of subspaces on each of which f is essential and let X_0 be their intersection. If f is inessential on X_0 then f is homotopic to a transformation g of X_0 into S such that $g(X_0)$ is a proper subset of S. Since S has the neighborhood extension property we may suppose that g has already been extended to an open set U which contains X_0. Moreover, H is normal. Hence, because we can find a neighborhood of X_0 whose closure is contained in U, we may assume that g is defined on $\overline{U}$. Then, by (2.2), since $f \sim g$ on X_0, there exists a neighborhood V of X_0 contained in U such that f is homotopic to g on V. In virtue of the fact that S is a Hausdorff space the set $g(X_0)$ is closed and in virtue of the normality of S there is an open set W which contains $g(X_0)$ and such that $S-W$ is not vacuous. For each $x \in X_0$ let P_x be a neighborhood of x with $P_x \subset V$ and $g(P_x) \subset W$. Let Q be the union of all the sets P_x. Then Q is an open set containing X_0, $Q \subset V$ and $g(Q) \subset W$. The open sets $[H-X]$ cover the closed set $H-Q$ and hence from the compactness of H and the fact that $[X]$ is an ordered family we conclude that $H-Q$ is contained in some single set $H-X'$. Thus $X' \subset Q$. But $f \sim g$ on V and so on X' and clearly $g(X')$ is a proper subset of S. This contradiction completes the proof.

This is a generalization of a result of G. T. Whyburn's [9, p. 222]. The next definition (though not the nomenclature) and the result which follows are due to W. Hurewicz [3, p. 94]. Let $f: X \rightarrow S$ and let Y be a subspace such that f admits an extension to any closed subset of Y but not to Y itself. Then Y is said to be an *essential membrane* for f. Only trivial changes are necessary to modify Hurewicz's proof so that we get:

(2.6) *Any mapping of a subset X of H into S which cannot be extended to a mapping of H into S admits an essential membrane.*

The next result is due to S. Eilenberg [2, p. 164].

(2.7) *If the subspace X admits an irreducibly essential mapping into an n-sphere then* (a) *if $n=0$, X is a 0-sphere and* (b) *if n is positive then X is a continuum.*

3. **Connectivity and the sets T_S.** In this section we shall be concerned with a generalization of the notion of connectedness and sets that are "trivial" relative to mappings into S. Most of our results follow rather easily from classical theorems if we admit that H is metric.

A topological space R is said to be *S-connected* provided that every mapping of R into S is inessential on each compact subset of R. By an *S-continuum* (or a C_S) is meant a compact S-connected space.

Here and later we shall be interested mainly in the situation in which S is an n-sphere. In this case we write *n-connected*, *n-continuum* and C_n respectively. It is easy to see that a 0-connected set is connected in the usual sense and conversely, under relatively mild conditions. Also if H is metric and locally connected than it is 1-connected if and only if it is unicoherent, a fact first proved by Borsuk. A more general situation was considered by Eilenberg (see [9]).

(3.1) *The property of being an S-connected subspace of H is inductive.*

Proof. Let $[X]$ be an ordered family of S-continua of H and let X_0 denote the intersection of these sets. Let $f: X_0 \to S$. We may assume that f has been extended to a neighborhood U of X_0. From the compactness of H and the fact that the collection $[X]$ is ordered we conclude that some X' of $[X]$ is contained in U. In virtue of the fact that X' is a C_S we see that f is inessential on X' and hence on X.

For $S=S_0$ this result is classical. For $S=S_1$ see Whyburn [9, p. 222]. From (3.1) and the Brouwer induction theorem we conclude, for example, that if H is a C_S, then any subspace admits an S-continuum irreducible about it.

A compact Hausdorff space is said to be a T_S provided that every one of its closed sets is a C_S. In other words a T_S is hereditarily a C_S. Clearly a closed subset of H is a T_0 if and only if it is a point.

(3.2) *Any T_n is a T_{n+1}.*

Proof. Let Y be a T_n and let X be a closed subset of Y. Suppose that f is a mapping of X into $S_{n+1}: y_1^2 + \cdots + y_{n+2}^2 = 1$. Decompose S_{n+1} into two $(n+1)$-cells A_1 and A_2 defined by $y_{n+2} \geqq 0$ and $y_{n+2} \leqq 0$ respectively. Clearly $f(X)$ covers S_{n+1} or else f is null-homotopic. Let $B_i = f^{-1}(A_i)$, $C = B_1 \cdot B_2$, so that C is the inverse under f of the n-sphere $D = A_1 \cdot A_2$. Let $g = f | C$. We know that g is not essential. Then there exists a mapping $g': C \times (01) \to D$ such that $g'(x, 0) = g(x)$ and $g'(x, 1) = \bar{y}$, a point of D. We may extend g' (retaining the notation) to $B_1 \times (0) + B_1 \times (1)$ by setting $g'(x, 0) = g(x)$, $g'(x, 1) = \bar{y}$ on B_1. Then g' transforms $B_1 \times (0) + C \times (01) + B_1 \times (1)$ into the $(n+1)$-cell A_1 and so may be extended to a mapping h_1 of $B_1 \times (01)$ into A_1 by Tietze's theorem [3, p. 82]. Similarly we secure a mapping h_2 of $B_2 \times (01)$ such that $h_1 = h_2$ on the common part of $B_1 \times (01)$ and $B_2 \times (01)$. We may then combine h_1 and h_2 to get a transformation h of $X \times (01)$ into S_{n+1} such that $h(x, 0) = f(x)$ and $h(x, 1) = \bar{y}$ for all $x \in X$. Accordingly f is not essential. Thus Y is a T_{n+1}.

A subspace will be termed a B_S provided that it is not cut by any T_S and is maximal relative to this property. For this notion and (3.3) see G. T. Whyburn [8].

(3.3) *If X is a subspace not cut by any T_S then X is contained in a B_S. Each B_S is a 0-continuum and the intersection of any distinct pair of them is a set T_S.*

(3.4) *For any B_n there exists a sequence $B_0 \supset B_1 \supset \cdots \supset B_n$.*

This result follows at once from (3.2).

(3.5) *If the subspace Y is the union of a countable collection of sets T_n then Y is a T_{n+1}.*

Proof. Let X be a subspace contained in Y and let $f: X \to S_{n+1}$. Since any subspace contained in a T_n is also a T_n we may set $X = X_1 + X_2 + X_3 + \cdots$, where each X_i is a T_n. We may assume that f has been extended to a neighborhood U of X in virtue of the fact that, as an ANR, X has the neighborhood extension property. There is a neighborhood U_1 such that $Y_1 = X_1 \subset U_1 \subset \overline{U}_1 \subset U$ such that $f \sim 0$ on $\overline{U}_1$, by (2.3) and (3.2). Let $Y_2 = \overline{U}_1 + X_2$. We may assume that $f(Y_2)$ covers S_{n+1} since otherwise f would be null-homotopic on Y_2. Let S_{n+1} be given by $y_1^2 + \cdots + y_{n+2}^2 = 1$ and define g as the transformation which takes $\overline{U}_1$ into the point $\bar{y} = (0, 0, \cdots, 0, 1)$. Now g is null-homotopic on $\overline{U}_1$ so that by the reflexivity of homotopy we have $f \sim g$ on this set. Hence by (2.1) we may extend g to a mapping $h: Y_2 \to S_{n+1}$ such that $f \sim g$ on Y_2. Let S_n be defined as the subset of S_{n+1} on which $y_{n+2} = 0$. Then $h^{-1}(S_n)$ lies wholly in $X_2 - \overline{U}_1$. Now if f were essential on Y_2 then the transformation h regarded as a mapping of $h^{-1}(S_n)$ into S_n would be essential by the argument given in the proof of (3.2). This is impossible since $h^{-1}(S_n)$ lies in the T_n, $X_2 - \overline{U}_1$. Hence we see that f is null-homotopic on Y_2. Accordingly there is a neighborhood U_2 of Y_2 such that $\overline{U}_2$ is contained in U and f is inessential on U_2. In this way we secure for each integer i a neighborhood U_i for which

$$Y_i = \overline{U}_{i-1} + X_i \subset U_i \quad \text{and} \quad f \sim 0 \quad \text{on} \quad Y_i = \overline{U}_i + X_{i+1} \subset U.$$

Since the open sets U_i cover the compact set X and in addition form an increasing family it follows that X lies in some set U_j. Thus f is null-homotopic on X and so X is a T_{n+1}.

We terminate this section with some additional theorems on n-continua.

(3.6) *A retract of a C_n is a C_n.*

Proof. Suppose that H is an n-continuum and let $r: H \to X$ be a retraction. Then X is a subspace so let $f: X \to S_n$ and set $g = fr$ so that g transforms H into the n-sphere. Then g is inessential. But $g | X = f$ so that f is also inessential.

It is easy to see that a Tychonoff cube is a C_n for any n. Indeed it is clear that we have:

(3.7) *If a subspace can be deformed over itself to a point then it is a C_n.*

It is also clear that an n-cell is a T_n. For it is a C_n and a mapping of any of its closed sets into S_n can be extended by a theorem of Hurewicz [3, p. 83].

Also it is of interest to observe the difference between a C_n and a γ^n-continuum [8]. In virtue of a result of H. Hopf [13] it is clear that S_3 is not a C_2 while it is a γ^2-continuum.

We say that a subspace X of H is a *divisor* of H if $H \times (0) + X \times (01) + H \times (1)$ is a retract of $H \times (01)$.

(3.8) *If the subspace X is both a divisor of H and a C_n then H is a C_n.*

Proof. Let f be a mapping of H into the n-sphere and let $g = f | X$. Then g is null-homotopic and so there exists a transformation $h: X \times (01) \to S_n$ such that $h(x, 0) = g(x)$ and $h(x, 1) = \bar{y} \in S_n$. We may extend h (conserving the notation) to a mapping of $K = H \times (0) + X \times (01) + H \times (1)$ into the n-sphere by letting

$h=f$ on $H\times(0)$ and $h=\bar{y}$ on $H\times(1)$. Let r retract $H\times(01)$ into K. We may then set $k=hr$ so that k maps $H\times(01)$ into S_n and $k=h$ on $H\times(0)$ and $k=\bar{y}$ on $H\times(1)$.

We return to the notion of a divisor in a later paper. The concept is a modification of one introduced by Lusternik and Schnirelmann [5, p. 40]. The result (3.8) is introduced here since, in its contra-positive form, it is helpful in examples.

(3.9) *If $f:X\rightarrow S_n$ is irreducibly essential on the subspace X and Y is an essential membrane for f then Y is a C_0.*

Proof. Let n be positive and suppose that $Y=P+Q$ where P and Q are disjoint closed subsets of Y. In virtue of (2.7) we may suppose that $X\subset P$ and then let $\bar{f}$ be an extension of f mapping P into S_n. It is then trivial that $\bar{f}$ admits an extension to $P+Q$, a contradiction. A similar argument is valid if $n=0$.

4. **Extension sets.** By an *extension set of order n of H* (or a J_n) will be meant a subspace M such that for each closed set X each mapping of $M\cdot X$ into the n-sphere can be extended to a mapping of X into the n-sphere. Here we shall prove, for example, that the intersection of extension sets is an extension set and that the sets B_n are extension sets. With the assumption that H is of type V we are able to show that a J_n is a J_{n+1}. It is clear that an extension set is a topological invariant of H and that any point of H is an extension set. Many of our results are generalizations of well known results of G. T. Whyburn and W. L. Ayres (see [9]).

(4.1) *The space H and each T_n-set is an extension set of order n.*

Proof. If M is a T_n and X is a subspace then any transformation f of $M\cdot X$ into S_n is inessential and hence by (2.1) may be extended to a mapping of X into S_n since the trivial mapping may always be extended.

(4.2) *If $[M]$ is an arbitrary collection of sets J_n then their intersection is also a J_n.*

Proof. Let X be a subspace and f a transformation of $M_0\cdot X$ into the n-sphere, where M_0 is the intersection of all the sets in $[M]$. We may extend f (keeping the same notation) to a neighborhood U of $M_0\cdot X$ in virtue of the fact that S_n has the neighborhood extension property. In view of the compactness of H and the fact that the sets $[M]$ are closed it follows that there exists (by the Borel theorem) a finite family $M_1, M_2, \cdots, M_p$ such that $M_1\cdot M_2\cdots M_p\cdot X\subset U$. Let $g_1=f|M_1\cdot M_2\cdots M_p\cdot X$. Now M_1 is an extension set and so g_1 may be extended to a mapping $g_2:M_2\cdots M_p\cdot X\rightarrow S_n$. Similarly g_2 may be extended to a mapping g_3 of the set $M_3\cdots M_p\cdot X$ into the n-sphere. Continuing in this way we arrive finally at a mapping g of X

into S_n which is an extension of f. Accordingly we have shown that M_0 is an extension set of order n.

Let M be a J_0 lying in the Peano space H and let C be an arc having only its end points a and b in M. Now S_0 consists of the point -1 and the point $+1$ of the real line. Define $f(a)=-1$ and $f(b)=+1$. Then obviously f cannot be extended to C since C is connected.

At this point it is perhaps well to give some examples. Let L be the interval from $(0, -1)$ to $(0, 1)$ in the plane and let W be the curve $y=\sin x^{-1}$ for $0<x\leqq 1$. Then H is the union of the sets W and L. The set L is a J_n for all non-negative integers n.

As another example let L be the unit interval and C the Cantor set on L. At each point c of C let $I(c)$ be an interval of unit length erected perpendicular to L and above L. Finally let H be the Cartesian product of (01) with the set composed of L and all the sets $I(c)$. Then $L\times(01)$ is a J_n for all positive integers n. It is easy to see that $L\times(01)$ is not a J_0.

The next example I owe to J. W. Tukey. Let C_1 and C_2 be concentric 1-spheres of radii 1 and 2 respectively. Let H be the closed ring they determine. For each point z of C_1 let $R(z)$ be that part of the ray through the center of C_1 which lies in H. We set up a topology in H as follows: (a) If z is a point of C_1 then a neighborhood of z will consist of an open arc of C_1 containing z together with all sets $R(z')$ for all z' in the open arc with the exception that on $R(z)$ there may have been deleted any closed interval not containing the point z. (b) If x is a point of $R(z)$ distinct from z then a neighborhood of x will be an open interval of $R(z)$ containing x. Then H is a compact connected locally connected Hausdorff space but is not separable. The set C_1 is a J_n for all positive n but is not a J_0.

(4.3) *If M is a J_n and X is a C_n then $M\cdot X$ is a C_n.*

Proof. Let f map $M\cdot X$ into the n-sphere and let $\bar{f}$ be an extension of f transforming X into the n-sphere. Since $\bar{f}$ is inessential on X it is manifestly inessential on X. But $f=\bar{f}|M\cdot X$ and so f is inessential. Accordingly $M\cdot X$ is a C_n.

This result carries with it the conclusion that if H is a C_n then so also is each J_n. For any set A contained in H let $\Delta_n(A)$ be the intersection of all sets J_n which contain A. Since H is a J_n the set $\Delta_n(A)$ is always defined. By (4.2) it follows that:

(4.4) *For any set $A\subset H$ the set $\Delta_n(A)$ is a J_n.*

It is not hard to see that if H is a Peano space then the J_0-sets and A-sets are identical as are B_0-sets and cyclic elements. If we omit the condition of local connectedness then each A-set is a J_0 but the converse is false as may be seen by simple examples. (In this connection see (5.4).) Also here the B_0-sets are identical with the E_0-sets.

We wish to show that any B_n is a J_n. We begin with:

(4.5) *If the subspace X is not cut by any T_n then $\Delta_n(X)$ also has this property.*

Proof. Let Z be a T_n cutting $\Delta_n(X)=M$ so that $M-Z=U+V$ where U and V are mutually separated. Now X is not a subset of Z since otherwise X would be a T_n and certainly some closed subset of X cuts X. We may then suppose that $X-Z\subset U$ so that U is not void. Let K be a subspace, $N=U+Z$ and let f be a mapping of $N\cdot K$ into the n-sphere. Now $K\cdot M=K\cdot N+K\cdot(V+Z)$ and we have $(K\cdot N)\cdot K\cdot(V+Z)=K\cdot Z$ and this set is a T_n. Thus $f|K\cdot Z$ is null-homotopic and so by (2.1) may be extended to a mapping of $K\cdot(V+Z)$ into S_n. We thus arrive at an extension of f, $\bar{f}:K\cdot M\rightarrow S_n$. From (4.4) it follows that $\bar{f}$ may be extended to a transformation of K into the n-sphere. Thus N is a J_n which contains X and does not meet V. Accordingly $V=0$ and the proof is complete.

(4.6) *Each set B_n is also a set J_n.*

Proof. Let B be a B_n so that B is not cut by any T_n and is maximal relative to this property. Then $\Delta_n(B)$ is not cut by any T_n and $B\subset\Delta_n(B)$. Hence $B=\Delta_n(B)$ and so is a J_n by (4.4).

At this point it seems worthwhile to comment on the fact that, although defined by quite different devices, the sets B_n are in a sense the nontrivial minimal sets of type J_n. Thus (4.6) lends a certain uniformity to the structural theory we develop.

(4.7) *If H is a space of type V then each J_n is a J_{n+1}.*

Proof. Let M be an extension set of order n, Y a subspace and f a transformation of $M\cdot Y$ into the $(n+1)$-sphere. As usual we suppose that S_{n+1} is given by $y_1^2+\cdots+y_{n+2}^2=1$ so that the sets A_1 and A_2 defined by $y_{n+2}\geqq 0$ and $y_{n+2}\leqq 0$ are $(n+1)$-cells meeting in an S_n. If $f(M\cdot Y)$ does not cover S_{n+1} then f is null-homotopic and so may be extended to Y by (2.1). We may thus suppose that $f(M\cdot Y)=S_{n+1}$ and let Q_i be the inverse of A_i under f so that $M\cdot Y=Q_1+Q_2$. Let $R=Q_1\cdot Q_2$. Since H is of type V (see the first part of section two) we may write $H=H_1+H_2$ where $Q_i\subset H_i$, $H_1\cdot H_2\cdot M\cdot Y=R$, with H_1 and H_2 closed. Put $Y_i=H_i\cdot Y$ so that we have $Q_i\subset Y_i$, $Y_1\cdot Y_2\cdot M\cdot Y=R$ and $Y=Y_1+Y_2$. Now let $g=f|R$ so that g maps R into S_n. Since M is a J_n we may extend g to $\bar{g}: Y_1\cdot Y_2\rightarrow S_n$. Let $f_1=f|Q_1$ and $f_1=\bar{g}|Y_1\cdot Y_2$ so that f_1 is a mapping of $Q_1+Y_1\cdot Y_2$ into A_1. By Tietze's extension theorem there is an extension of f_1, $h_1: Y_1\rightarrow A_1$. Similarly we construct a transformation $h_2: Y_2\rightarrow A_2$ for which we have $h_1=h_2$ on $Y_1\cdot Y_2$. If we combine h_1 and h_2 we secure a mapping $h: Y\rightarrow S_n$ such that $h|M\cdot Y=f$. Thus M is a J_{n+1}.

(4.8) *If H is of type V and $A\subset H$ then*

$$\Delta_0(A)\supset\Delta_1(A)\supset\cdots\supset A.$$

The proof of (4.8) is immediate from (4.7). This result should be compared with (3.4).

(4.9) *Let M be a J_n and let $f:X\rightarrow S_n$ where X is a closed subset of M. If N is an essential membrane for f then $N\subset M$.*

Proof. By definition f can be extended to any proper subspace of N. If N is not contained in M then f can be extended to $M\cdot N$ and then to N since M is an extension set.

In an entirely analogous manner it is possible to prove that:

(4.10) *If M is a J_n, X a subset of M and N a C_n irreducible about X, then $N\subset M$.*

Let P be a property admissible for subspaces. A closed subset X of H will be termed a *P-endelement* provided that for each neighborhood U of X there is a neighborhood V of X contained in U such that $F(V)=\overline{V}-V$ has property P. If X is a point we say that it is a *P-endpoint.*

(4.11) *For any property P the property of being a P-endelement is inductive.*

The proof of (4.11) is immediate in view of the compactness of H.

(4.12) *Any T_n-endelement of H is a J_n.*

Proof. Let M be a T_n-endelement, Y a subspace and f a mapping of $M\cdot Y$ into the n-sphere. We may suppose that f has already been extended to a neighborhood U of $M\cdot Y$ and indeed we may suppose that the extension $\bar{f}$ is defined on $\overline{U}$. We may also assume that Y does not lie wholly in $\overline{U}$ and that $F(U)$ meets Y. Now M and $Y\cdot F(U)$ are disjoint closed sets. Let W be a neighborhood of $Y\cdot F(U)$ whose closure does not meet the set M. Then $P=H-(\overline{W}+(Y-U))$ is an open set containing M since

$$\begin{aligned} M\cdot P &= (M-\overline{W})\cdot((M-Y)+U)) \\ &= (M-Y)+M\cdot U\supset (M-Y)+M\cdot Y\cdot U = M. \end{aligned}$$

Now P contains a neighborhood V of M such that $F(V)$ is a T_n and we may even suppose that $\overline{V}$ is contained in P. Then $Y\cdot F(V)$ is contained in U. Let $g=\bar{f}|\ Y\cdot\overline{U}\cdot\overline{V}$. Since $M\cdot Y\subset Y\cdot\overline{U}\cdot\overline{V}$ it follows that g is an extension of f. Also

$$Y = Y\cdot\overline{U}\cdot\overline{V}+(Y-U\cdot V) \quad\text{and}\quad Y\cdot\overline{U}\cdot\overline{V}\cdot(Y-U\cdot V)\subset Y\cdot F(V).$$

Since $F(V)$ is a T_n the mapping $g|\ Y\cdot F(V)$ may be extended to a transformation $h:(Y-U\cdot V)\rightarrow S_n$. Combining g and h we get an extension of f mapping Y into S_n.

As a corollary to (4.12) we get:

(4.13) *If H is a C_n then any T_n-endelement is a C_n.*

5. **Further properties of extension sets.** It follows at once from the definition of the sets J_n that:

(5.1) *For any subset* $Z \subset H$ *we have* $\Delta_n(A) = \Delta_n(\overline{A})$.

(5.2) *If* M *is a* J_n, Z *a* T_n *and* $M - Z = M_1 + M_2$ *is a separation then* $M_i + Z$ *is a* J_n.

The proof of this is similar to that of (4.5).

(5.3) *If* x *and* y *are points of* H *then neither* x *nor* y *is a cut point of* $\Delta_n(x+y)$ *if this set is a continuum.*

This is a corollary to (5.2). In this connection see Ayres [1]. Our next result is highly analogous to a well known proposition concerning A-sets. Thus, in a Peano space, the complement of each A-set is the union of a null-sequence of pair-wise disjoint open sets (the components of its complement) each of which has a T_0 for its boundary.

(5.4) *Let* N *be a subspace the complement of which is the union of a collection of pair-wise disjoint open sets whose boundaries are sets* T_n. *Then* N *is a* J_n.

Proof. Let $H - N$ be covered by the collection $[U]$ of pair-wise disjoint open sets such that $F(U)$ is a T_n for each $U \in [U]$. Let f transform $N \cdot Y$ into the n-sphere, Y being a subspace. We may extend f to a mapping $g: V \rightarrow S_n$ where V is a neighborhood of $Y \cdot N$. In view of the normality of H we can find a neighborhood W of $N \cdot Y$ such that $\overline{W} \subset V$. We have

$$Y = Y \cdot N + \sum U \cdot Y = Y \cdot W + \sum U \cdot Y.$$

The sets $Y \cdot W$, $Y \cdot U$ are open in Y and Y is compact. Hence there is a finite family $U_1, U_2, \cdots, U_p$ such that the sets $Y \cdot W$, $Y \cdot U_i$ cover Y. Let $Z_1 = Y \cdot \overline{W} - U_1$ and $Y_1 = Z_1 + \overline{U}_1$. Then $Z_1 \cdot \overline{U}_1$ is a subset of $Y \cdot \overline{W} \cdot F(U_1)$ and so is a T_n. The mapping $g | Z_1 \cdot \overline{U}_1$ can be extended to a mapping of $\overline{U}_1$ into S_n. We secure in this way a transformation g_1 of Z_1 into S_n which is an extension of g. Let $Z_2 = Y_1 - U_2$ and $Y_2 = Z_2 + \overline{U}_2$. Then $Z_2 \cdot \overline{U}_2$ is contained in $Y_1 \cdot F(U_2)$ and so is a T_n-set. Accordingly the mapping $g_1 | Z_2 \cdot \overline{U}_2$ may be extended to a transformation g_2 of Y_2 into S_n. By a continuation of this procedure we arrive finally at a mapping $g_p: Y_p \rightarrow S_n$ which is an extension of g_{p-1} and hence of f. But Y is a subset of Y_p and we are able to infer that N is an extension set.

(5.5) *If* X *is a subspace then each* T_n*-endpoint of* $\Delta_n(X)$ *is a* T_n*-endpoint of* X.

Proof. There is no loss of generality if we let H be the set $\Delta_n(X)$. Thus let p be a T_n-endpoint of H not in X so that there is a neighborhood U containing p which does not meet X. Then U contains a neighborhood V of p

for which $F(V)$ is a T_n-set. But then $H-V$ is a J_n (by (5.4)) which contains X but not p. This is a contradiction.

If we regard the sets J_n as "closed" sets then (5.5) states that any subspace "dense" in H contains all T_n-endpoints. In this connection see Ayres [1]. We remark that (5.5) remains valid if "T_n-endpoint" is replaced by "T_n-endelement."

(5.6) *If $M+N$ and $M\cdot N$ are J_n-sets then so also are M and N if they are subspaces.*

Proof. Let Y be a subspace and let f be a mapping of $Y\cdot M$ into the n-sphere. The cases in which $Y\cdot N=0$ or $Y\cdot M\cdot N=0$ are readily treated. The only other case is that in which $Y\cdot M\cdot N$ is not null. Then $f|Y\cdot M\cdot N$ can be extended to a mapping g of $Y\cdot N$ into S_n. Call this mapping g. If we combine f and g we get a transformation $h: Y\cdot(M+N)\rightarrow S_n$. This mapping may be extended to Y by our hypothesis.

(5.7) *If X is a C_0 then $\Delta(X)$ is a C_0.*

Proof. For if not, then we would have a separation as in (5.2) with $Z=0$. If X is a C_0 it must lie in one of the sets M_1 or M_2, say M_1. But M_1 is a J_n by (5.2).

(5.8) *Let Z be the set of all points of H that are not T_n-endpoints. Then, if H is a C_n, the set Z is n-connected.*

Proof. If not let $f: Z\rightarrow S_n$ which is essential on the subspace X. Since H is an n-continuum and thus f cannot be extended to H it follows by (2.6) that f admits an essential membrane Y. Suppose that the point p of Y is a T_n-endpoint. Then there exists a neighborhood V of p such that $F(V)$ is a T_n and $V\subset H-X$, since X is closed and p is not in X. Now $X\subset Y-V$ and since Y is an essential membrane f can be extended to $\bar{f}:(Y-V)\rightarrow S_n$. Since $Y=Y\cdot\overline{V}+(Y-V)$ and $Y\cdot\overline{V}\cdot(Y-V)$ is a T_n it follows that $\bar{f}|Y\cdot F(V)$ can be extended to $Y\cdot\overline{V}$. Thus f can be extended to Y, a contradiction. Hence Y must be contained in Z and so f is defined on Y. This is impossible. Hence S is n-connected. For $n=0$ this result is well known.

Bibliography

1. W. L. Ayres, *Concerning the arc-curves and basic sets of a continuous curve*, Trans. Amer. Math. Soc. vol. 30 (1928) pp. 567–578.

2. S. Eilenberg, *Sur les transformations d'espaces métriques en circonférence*, Fund. Math. vol. 24 (1935) pp. 160–176.

3. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series, vol. 4, 1941.

4. S. Lefschetz, *Topics in topology*, Annals of Mathematics Studies, no. 10, 1942.

5. L. Lusternik and L. Schnirelmann, *Méthodes topologiques*, Paris, 1934.

6. J. Rozanska, *Über stetige Abbildungen eines Elementes*, Fund. Math. vol. 28 (1937) pp. 219–232.

7. A. D. Wallace, *Monotone transformations*, Duke Math. J. vol. 9 (1942) pp. 487–506.

8. G. T. Whyburn, *Cyclic elements of higher order*, Amer. J. Math. vol. 56 (1934) pp. 133–146.

9. ———, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

10. R. L. Wilder, *Point sets in three and higher dimensions and their investigation by means of a unified analysis situs*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 649–692.

11. ———, *The sphere in topology*, Amer. Math. Soc. Semicentennial Publications, vol. 2, 1938, pp. 136–184.

12. C. Borsuk, *Über Rand- und Kernpunkte kompakter metrischer Räume*, Ergebnisse eines mathematischen Kolloquiums, Vienna, 1933, pp. 28–32.

13. H. Hopf, *Über die Abbildungen von Sphären*, Fund. Math. vol. 25 (1935) pp. 427–440.

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